A model of the term structure of interest rates based on Lévy fields

Sergio Albeverio

Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn, Germany; SFB 611, Univ. Bonn, Germany; CERFIM (Locarno); Acc. Arch. (USI), Switzerland; BiBoS, Univ. Bielefeld, Germany

E-mail: albeverio@uni-bonn.de

Eugene Lytvynov

Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn, Germany; SFB 611, Univ. Bonn, Germany; BiBoS, Univ. Bielefeld, Germany

E-mail: lytvynov@wiener.iam.uni-bonn.de

Andrea Mahnig

Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, D-53115 Bonn

E-mail: andrea.mahnig@gmx.de

Abstract

An extension of the Heath–Jarrow–Morton model for the development of instantaneous forward interest rates with deterministic coefficients and Gaussian as well as Lévy field noise terms is given. In the special case where the Lévy field is absent, one recovers a model discussed by D.P. Kennedy.

Keywords: Term structure of interest rates; Lévy fields; HJM-model; Kennedy model

JEL Classification: E43

2000 *AMS Mathematics Subject Classification*: 91B28, 60J75, 60H15, 60G51

1 Introduction

Heath, Jarrow, and Morton (1992) (see also Heath et al. (1990)) proposed a model of interest rates and their associated bond prices in which the price at time s of a bond paying one unit at time $t \geq s$ is given by

$$P_{s,t} = \exp\left[-\int_s^t F_{s,u}\,du\right], \tag{1.1}$$

where $F_{s,t,}$, $0 \le s \le t$, is called the *instantaneous forward rate*, or just the *forward rate*. Then

$$R_s{:=}F_{s,s}, \qquad s \ge 0, \tag{1.2}$$

is called the *instantaneous spot rate*, or just the *spot rate*. One also defines the *discounted bond-price process* as

$$Z_{s,t}{:=}P_{s,t}\,\exp\left[-\int_0^s R_u\,du\right]. \tag{1.3}$$

In the Heath–Jarrow–Morton (HJM) model, the forward rates are supposed to satisfy the stochastic differential equations

$$dF_{s,t} = \alpha(s,t)\,ds + \sum_{i=1}^{m} \beta_i(s,t)\,dW_s^i, \tag{1.4}$$

where $W^1, \ldots, W^m$ are independent standard Brownian motions and $\alpha(s,t)$ and $\beta_i(s,t)$ are processes adapted to the natural filtration of the Brownian motions. This model was, in fact, an extension of the earlier work by Ho et al. (1986).

Kennedy (1994) (see also Kennedy (1997)), while following the approach of modeling the instantaneous forward rates, considered the case where $\{F_{s,t},\ 0 \le s \le t < \infty\}$ is a continuous Gaussian random field which has independent increments in the s-direction, that is, in the direction of evolution of 'real' time. This framework includes the HJM model in the case where the coefficients $\alpha(s,t)$ and $\beta_i(s,t)$ in (1.4) are deterministic. An important example of application of the Kennedy model is the case where the forward rates are given by $F_{s,t} = \mu_{s,t} + X_{s,t}$ with $\mu_{s,t}$ being deterministic and $X_{s,t}$ a Brownian sheet (see, e.g., Adler (1981) for this concept). In the latter case, $F_{s,t}$ has independent increments also in the t direction. Furthermore, this may be intuitively thought of as the situation of (1.4) driven by an uncountably infinite number of Brownian motion. Kennedy (1994) gave a simple characterization of the discounted bond-price process to be a martingale. In particular, he showed that the latter is true if and only if the expectation $\mu_{s,t}$ of $F_{s,t}$, $0 \le s \le t < \infty$, satisfies a simple relation.

In Björk et al. (1997a, b) (see also Björk et al. (1999)), it was pointed out that, in many cases observed empirically, the interest rate trajectories do not look like diffusion processes, but rather as diffusions and jumps, or even like pure jump processes. Therefore, one needs to introduce a jump part in the description of interest rates. The authors of these papers considered the case where the forward rate process $\{F_{s,t},\ 0 \le s \le t\}$ is driven by a general marked point process as well as by a Wiener process (Björk et al. (1997b)), or by a rather general Lévy process (Björk et al. (1997a)), and the maturity time $t \ge 0$ is a continuous parameter of the model. In particular, an equivalence condition was given for a given probability measure to be a local martingale measure, i.e., for the discounted bond-price process $\{Z_{s,t},\ 0 \le s \le t\}$ to be a local martingale

for each $t \geq 0$ (Propositions 5.3, 5.5 in Björk et al. (1997a), see also Theorem 3.13 in Heath et al. (1992)). This condition, formulated in terms of the coefficients for the forward rate dynamics, generalizes the result of Heath et al. (1990) which was obtained for the diffusion case.

Other generalizations of the HJM model in which the forward rate process satisfies stochastic differential equations with an infinite number of independent standard Brownian motions (i.e., $m = \infty$ in (1.4)) were proposed in Yalovenko (1998), Kusuoka (2000), Popovici (2001), and Lütkebohmert (2002).

In the present paper, we follow the approach of Kennedy (1994, 1997), but suppose that the forward rates $\{F_{s,t},\, 0 \leq s \leq t < \infty\}$ are given by a Lévy field without a diffusion part. In particular, $\{F_{s,t}\}$ has independent increments in both s and t directions. Analogously to Kennedy (1994), we give, in this case, a characterization of the martingale measure. We also show that, under a slight additional condition on the Lévy measure of the field, it is possible to choose the initial term structure $\{\mu_{0,t},\, t \geq 0\}$ in such a way that the forward interest rates are a.s. non-negative. This, of course, was impossible to reach in the framework of the Gaussian model, which caused problems in some situations (see Sect. 1 of Kennedy (1997)). We then present two examples of application of our results: the cases where $F_{s,t}$ is a "Poisson sheet" (this case was discussed in Mahnig (2002)), respectively a "gamma sheet." Finally, we mention the possibility of unification of the approaches of Kennedy and of the present paper, by considering $F_{s,t}$ as a sum of a Gaussian field and an independent Lévy field, and thus having a process with a diffusion part as well as a jump part.

2 The model based on Lévy fields

Let $\mathcal{D}{:=}C_0^\infty(\mathbb{R}^2)$ denote the space of all real-valued infinitely differentiable functions on $\mathbb{R}^2$ with compact support. We equip $\mathcal{D}$ with the standard nuclear space topology, see, e.g., Berezansky et al. (1996). Then $\mathcal{D}$ is densely and continuously embedded into the real space $L^2(\mathbb{R}^2, dx\,dy)$. Let $\mathcal{D}'$ denote the dual space of $\mathcal{D}$ with respect to the "reference" space $L^2(\mathbb{R}^2, dx\,dy)$, i.e., the dual pairing between elements of $\mathcal{D}'$ and $\mathcal{D}$ is generated by the scalar product in $L^2(\mathbb{R}^2, dx\,dy)$. Thus, we get the standard (Gel'fand) triple

$$\mathcal{D}' \supset L^2(\mathbb{R}^2, dx\,dy) \supset \mathcal{D}.$$

We denote by $\langle\cdot,\cdot\rangle$ the dual pairing between elements of $\mathcal{D}'$ and $\mathcal{D}$. Let $\mathcal{C}(\mathcal{D}')$ denote the cylinder σ-algebra on $\mathcal{D}'$.

We define a centered Lévy noise measure as a probability measure ν on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$ whose Fourier transform is given by

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\,\nu(d\omega) = \exp\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}_+}(e^{i\tau\varphi(x,y)} - 1 - i\tau\varphi(x,y))\,\sigma(d\tau)\,dx\,dy\right], \qquad \varphi \in \mathcal{D} \quad (2.1)$$

(see, e.g., Ch. III, Sec. 4 in Gel'fand and Vilenkin (1964)). Here, $\mathbb{R}_+ = (0, +\infty)$ and σ is a positive measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, which is usually called the Lévy measure of the process. We suppose that σ satisfies the following condition:

$$\int_{\mathbb{R}_+} \tau^2 \, \sigma(d\tau) < \infty. \tag{2.2}$$

The existence of the measure ν follows from the Minlos theorem.

For any $\varphi \in \mathcal{D}$, we easily have

$$\int_{\mathcal{D}'} \langle \omega, \varphi \rangle^2 \, \nu(d\omega) = \int_{\mathbb{R}_+} \tau^2 \, \sigma(d\tau) \int_{\mathbb{R}^2} \varphi(x, y)^2 \, dx \, dy. \tag{2.3}$$

Thus, the mapping $I : L^2(\mathbb{R}^2, dx\, dy) \to L^2(\mathcal{D}', d\nu)$, $\mathrm{Dom}(I) = \mathcal{D}$, defined by

$$(I\varphi)(\omega) := \langle \omega, \varphi \rangle, \qquad \varphi \in \mathcal{D}, \ \omega \in \mathcal{D}',$$

may be extended by continuity to the whole $L^2(\mathbb{R}^2, dx\, dy)$. For each $f \in L^2(\mathbb{R}^2, dx\, dy)$, we set $\langle \cdot, f \rangle := If$. Thus, the random variable (r.v.) $\langle \omega, f \rangle$ is well-defined for ν-a.e. $\omega \in \mathcal{D}'$ and equality (2.3) holds for f replacing φ.

Let $\varkappa : [0, \infty)^2 \to [0, \infty)$ be a measurable function which is locally bounded. For each $s, t \geq 0$, we define the r.v. $X_{s,t}$ as follows:

$$X_{s,t}(\omega) := \langle \omega(x, y), \mathbf{1}_{[0,s]}(x) \mathbf{1}_{[0,t]}(y) \varkappa(x, y) \rangle, \qquad \nu\text{-a.e. } \omega \in \mathcal{D}' \ , \tag{2.4}$$

where x, y denote the variables in which the dualization is carried out. It follows from (2.1) that $X_{s,t}$ is centered and has independent increments in both the s and t directions.

We note that, in the case where $\varkappa(x, y) \equiv 1$, $\{X_{s,t}, \ 0 \leq s \leq t\}$ is a Lévy process for each fixed $t > 0$. Indeed, it follows from (2.1) that the Fourier transform of $X_{s,t}$ is given by

$$\int_{\mathcal{D}'} e^{i\lambda X_{s,t}(\omega)} \, \nu(d\omega) = \exp\left[st \int_{\mathbb{R}_+} (e^{i\tau\lambda} - 1 - i\tau\lambda) \, \sigma(d\tau) \right], \qquad \lambda \in \mathbb{R}.$$

In particular, the Lévy measure of the process $\{X_{s,t}, \ 0 \leq s \leq t\}$ is equal to $t\sigma$.

Let $F_{s,t}$ be the forward rate for date t at time s, $0 \leq s \leq t$. We suppose that

$$F_{s,t} = \mu_{s,t} + X_{s,t}, \qquad 0 \leq s \leq t, \tag{2.5}$$

where $\mu_{s,t}$ is deterministic and continuous in (s, t) on $\{(s, t) \in \mathbb{R}^2 : s \leq t\}$. The price at time s of a bond paying one unit at time $t \geq s$ is then given by (1.1). We note that the random variable $\int_s^t X_{s,u} \, du$ is ν-a.s. well-defined and

$$\int_s^t X_{s,u}(\omega) \, du = \langle \omega(x, y), \mathbf{1}_{[0,s]}(x) \mathbf{1}_{[0,t]}(y) \varkappa(x, y)(t - (s \vee y)) \rangle \quad \text{for } \nu\text{-a.e. } \omega \in \mathcal{D}' \ . \tag{2.6}$$

Indeed, for each $f \in L^2(\mathbb{R}^2, dx\, dy)$, we have by (2.3):

$$
\begin{aligned}
&\int_{\mathcal{D}'} \left(\int_s^t X_{s,u}(\omega)\, du \right) \langle \omega, f \rangle\, \nu(d\omega) \\
&\qquad = \int_s^t \int_{\mathcal{D}'} X_{s,u}(\omega) \langle \omega, f \rangle\, \nu(d\omega)\, du \\
&\qquad = \int_{\mathbb{R}_+} \tau^2\, \sigma(d\tau) \cdot \int_s^t \int_{\mathbb{R}^2} \mathbf{1}_{[0,s]}(x) \mathbf{1}_{[0,u]}(y) \varkappa(x,y) f(x,y)\, dx\, dy\, du \\
&\qquad = \int_{\mathbb{R}_+} \tau^2\, \sigma(d\tau) \cdot \int_{\mathbb{R}^2} \mathbf{1}_{[0,s]}(x) \left(\int_s^t \mathbf{1}_{[0,u]}(y)\, du \right) \varkappa(x,y) f(x,y)\, dx\, dy \\
&\qquad = \int_{\mathbb{R}_+} \tau^2\, \sigma(d\tau) \cdot \int_{\mathbb{R}^2} \mathbf{1}_{[0,s]}(x) \mathbf{1}_{[0,t]}(y) (t - (s \vee y)) \varkappa(x,y) f(x,y)\, dx\, dy,
\end{aligned}
\tag{2.7}
$$

which implies (2.6).

Analogously to (2.6), (2.7), we have:

$$
\int_0^s R_u\, du = \int_0^s \mu_{u,u}\, du + \langle \omega(x,y), \mathbf{1}_{[0,s]}(x) \mathbf{1}_{[0,s]}(y) \varkappa(x,y)(s - (x \vee y)) \rangle,
$$

where the spot rate R_u is defined by (1.2). Thus, the discounted bond-price process $Z_{s,t}$ is well-defined by (1.3).

We denote by $\mathcal{F}_s$, $s \geq 0$, the σ-algebra generated by the r.v.'s $F_{u,v}$, $0 \leq u \leq s$, $u \leq v$, which describes the information available at time s.

Theorem 2.1 *The following statements are equivalent*:
(a) *For each* $t \geq 0$, *the discounted bond-price process* $\{Z_{s,t},\ \mathcal{F}_s,\ 0 \leq s \leq t\}$ *is a martingale.*
(b)

$$
\mu_{s,t} = \mu_{0,t} + \int_0^t \int_0^s \int_{\mathbb{R}_+} \tau \varkappa(x,y) (1 - e^{-\tau \varkappa(x,y)(t - (x \vee y))})\, \sigma(d\tau)\, dx\, dy
$$

for all $s, t \geq 0$, $s \leq t$.
(c)

$$
P_{s,t} = \mathbb{E}(e^{-\int_s^t R_u\, du} \mid \mathcal{F}_s) \qquad \text{for all } s, t \geq 0,\ s \leq t.
$$

Proof. We first show the equivalence of (a) and (b). Absolutely analogously to the proof of Theorem 1.1 in Kennedy (1994), we conclude that (a) is equivalent to the following condition to hold:

$$
\int_{\mathcal{D}'} \exp\left(-\int_{s_1}^t (F_{s_1,u} - F_{s_2,u})\, du - \int_{s_2}^{s_1} (F_{u,u} - F_{s_2,u})\, du \right) d\nu = 1 \tag{2.8}
$$

for all $0 \le s_2 \le s_1 \le t$. By (2.4) and (2.5), (2.8) is equivalent to

$$\int_{\mathcal{D}'} \exp\bigg(- \int_{s_1}^{t} \langle \omega(x,y), \mathbf{1}_{[s_2,s_1]}(x)\mathbf{1}_{[0,u]}(y)\varkappa(x,y)\rangle\, du \\ - \int_{s_2}^{s_1} \langle \omega(x,y), \mathbf{1}_{[s_2,u]}(x)\mathbf{1}_{[0,u]}(y)\varkappa(x,y)\rangle\, du \bigg)\, \nu(d\omega) \\ = \exp\bigg(\int_{s_1}^{t} (\mu_{s_1,u} - \mu_{s_2,u})\, du + \int_{s_2}^{s_1} (\mu_{u,u} - \mu_{s_2,u})\, du \bigg) \tag{2.9}$$

for all $0 \le s_2 \le s_1 \le t$. Analogously to (2.6) and (2.7), we have

$$\int_{s_1}^{t} \langle \omega(x,y), \mathbf{1}_{[s_2,s_1]}(x)\mathbf{1}_{[0,u]}(y)\varkappa(x,y)\rangle\, du + \int_{s_2}^{s_1} \langle \omega(x,y), \mathbf{1}_{[s_2,u]}(x)\mathbf{1}_{[0,u]}(y)\varkappa(x,y)\rangle\, du \\ = \langle \omega(x,y), \mathbf{1}_{[s_2,s_1]}(x)\mathbf{1}_{[0,t]}(y)\varkappa(x,y)(t-(x\vee y))\rangle \qquad \text{for } \nu\text{-a.e. } \omega \in \mathcal{D}'.$$

Hence, it follows from (2.1) that condition (2.9) is equivalent to

$$\int_{\mathbb{R}_+} \int_0^t \int_{s_2}^{s_1} (e^{-\tau\varkappa(x,y)(t-(x\vee y))} - 1 + \tau\varkappa(x,y)(t-(x\vee y)))\, dx\, dy\, \sigma(d\tau) \\ = \int_{s_1}^{t} (\mu_{s_1,u} - \mu_{s_2,u})\, du + \int_{s_2}^{s_1} (\mu_{u,u} - \mu_{s_2,u})\, du \tag{2.10}$$

for all $0 \le s_2 \le s_1 \le t$. We remark that the function under the sign of integral on the left hand side of (2.10) is integrable. Indeed, let us set $C_t := \sup_{x,y\in[0,t]} \varkappa(x,y)$. Then, for all $\tau \in (0,1]$ and $x,y \in [0,t]$, we have

$$|e^{-\tau\varkappa(x,y)(t-(x\vee y))} - 1 + \tau\varkappa(x,y)(t-(x\vee y))| \\ \le \sum_{n=2}^{\infty} \frac{(\tau C_t t)^n}{n!} \le \tau^2 C_t^2 t^2 \exp(\tau C_t t) \le \tau^2 C_t^2 t^2 \exp(C_t t). \tag{2.11}$$

This, together with the fact that $\int_{(0,1]} \tau^2\, \sigma(d\tau) < \infty$, yields the integrability of the integrand on the left hand side of (2.10) on $(0,1] \times [0,t] \times [s_1,s_2]$. Furthermore, for $\tau \in (1,+\infty)$ and $x,y \in [0,t]$, we have

$$|e^{-\tau\varkappa(x,y)(t-(x\vee y))} - 1 + \tau\varkappa(x,y)(t-(x\vee y))| \le 1 + \tau t C_t.$$

This, together with the fact that $\int_{(1,+\infty)} \tau\, \sigma(d\tau) < \infty$, completes the proof of the integrability of the integrand on the left hand side of (2.10).

We now fix $t > 0$ and suppose, for a moment, that $\mu_{s,t}$ has the following form:

$$\mu_{s,t} = \mu_{0,t} + \int_0^t \int_0^s \Psi_t(x,y)\, dx\, dy, \tag{2.12}$$

where $\Psi_t(x,y)$ is an integrable function on $[0,t]^2$. Then,

$$\int_{s_1}^{t}(\mu_{s_1,u}-\mu_{s_2,u})\,du+\int_{s_2}^{s_1}(\mu_{u,u}-\mu_{s_2,u})\,du=\int_0^t\int_{s_2}^{s_1}\Psi_t(x,y)(t-(x\vee y))\,dx\,dy. \quad (2.13)$$

Comparing (2.13) with (2.10), we see that condition (2.10) is, at least formally, satisfied if $\Psi_t(x,y)$ has the form

$$\Psi_t(x,y)=(t-(x\vee y))^{-1}\int_{\mathbb{R}_+}(e^{-\tau\varkappa(x,y)(t-(x\vee y))}-1+\tau\varkappa(x,y)(t-(x\vee y)))\,\sigma(d\tau). \quad (2.14)$$

To show that this inserted into (2.12) gives indeed a solution of (2.10), we have to verify that the $\Psi_t(x,y)$ given by (2.14) is integrable on $[0,t]^2$. Analogously to (2.11), we get

$$\begin{aligned}\int_0^t\int_0^t\int_{(0,1]}&|(t-(x\vee y))^{-1}(e^{-\tau\varkappa(x,y)(t-(x\vee y))}-1+\tau\varkappa(x,y)(t-(x\vee y)))|\,\sigma(d\tau)\,dx\,dy\\ &\le\int_0^t\int_0^t\int_{(0,1]}\sum_{n=2}^{\infty}\frac{\tau^n\varkappa(x,y)^n(t-(x\vee y))^{n-1}}{n!}\,\sigma(d\tau)\,dx\,dy\\ &\le t^3C_t^2e^{tC_t}\int_{(0,1]}\tau^2\,\sigma(d\tau)<\infty. \end{aligned}\quad (2.15)$$

Next,

$$\begin{aligned}\int_0^t\int_0^t\int_{(1,+\infty)}&|(t-(x\vee y))^{-1}(e^{-\tau\varkappa(x,y)(t-(x\vee y))}-1+\tau\varkappa(x,y)(t-(x\vee y)))|\,\sigma(d\tau)\,dx\,dy\\ &\le\int_0^t\int_0^t\int_{(1,+\infty)}|(t-(x\vee y))^{-1}(e^{-\tau\varkappa(x,y)(t-(x\vee y))}-1)|\,\sigma(d\tau)\,dx\,dy\\ &\quad+t^2C_t\int_{(1,+\infty)}\tau\,\sigma(d\tau)\\ &\le 2t^2C_t\int_{(1,+\infty)}\tau\,\sigma(d\tau), \end{aligned}\quad (2.16)$$

where we used the estimate: $1-e^{-\alpha}\le\alpha$ for all $\alpha\ge 0$. Thus, by (2.12) and (2.14)–(2.16), statement (a) holds for

$$\mu_{s,t}=\mu_{0,t}+\int_0^t\int_0^s\int_{\mathbb{R}_+}(t-(x\vee y))^{-1}(e^{-\tau\varkappa(x,y)(t-(x\vee y))}-1+\tau\varkappa(x,y)(t-(x\vee y)))\,\sigma(d\tau)\,dx\,dy. \quad (2.17)$$

Let us now suppose that (a), or equivalently (2.10), holds. We set $s_2=0$ and $s_1=s$. Then, (2.10) takes the following form:

$$\int_0^t\int_0^s\int_{\mathbb{R}_+}(e^{-\tau\varkappa(x,y)(t-(x\vee y))}-1+\tau\varkappa(x,y)(t-(x\vee y)))\,\sigma(d\tau)\,dx\,dy$$

$$= \int_s^t (\mu_{s,u} - \mu_{0,u})\, du + \int_0^s (\mu_{u,u} - \mu_{0,u})\, du. \tag{2.18}$$

Differentiating (2.18) in t yields for $s \le t$:

$$\mu_{s,t} - \mu_{0,t} = \int_0^t \int_0^s \int_{\mathbb{R}_+} \tau\varkappa(x,y)(1 - e^{-\tau\varkappa(x,y)(t-(x\vee y))})\,\sigma(d\tau)\,dx\,dy. \tag{2.19}$$

That the integral on the right hand side of (2.19) is finite may be shown analogously to (2.15), (2.16).

Since for the $\mu_{s,t}$ given by (2.17) statement (a) holds, this $\mu_{s,t}$ also satisfies (2.19). Therefore,

$$\begin{aligned}\int_0^t \int_0^s \int_{\mathbb{R}_+} &(t-(x\vee y))^{-1}(e^{-\tau\varkappa(x,y)(t-(x\vee y))} - 1 + \tau\varkappa(x,y)(t-(x\vee y)))\,\sigma(d\tau)\,dx\,dy \\ &= \int_0^t \int_0^s \int_{\mathbb{R}_+} \tau\varkappa(x,y)(1 - e^{-\tau\varkappa(x,y)(t-(x\vee y))})\,\sigma(d\tau)\,dx\,dy,\end{aligned}$$

which implies the equivalence of (a) and (b).

Let us now show the equivalence of (b) and (c). Analogously to Kennedy (1994), we conclude that (c) is equivalent to

$$\int_{\mathcal{D}'} \exp\left(- \int_s^t (F_{u,u} - F_{s,u}) \right) d\nu = 1 \tag{2.20}$$

for all $s,t \ge 0$, $s \le t$. Analogously to the above, we see that (2.20) is, in turn, equivalent to

$$\begin{aligned}\int_0^t \int_s^t \int_{\mathbb{R}_+} &(e^{-\tau\varkappa(x,y)(t-(x\vee y))} - 1 + \tau\varkappa(x,y)(t-(x\vee y)))\,\sigma(d\tau)\,dx\,dy \\ &= \int_s^t (\mu_{u,u} - \mu_{s,u})\,du \end{aligned}\tag{2.21}$$

for all $s,t \ge 0$, $s \le t$. Setting in (2.10) $s_2 = s$ and $s_1 = t$, we see that (2.21) is a special case of (2.10), so that (b) implies (c). To show the inverse conclusion, we follow Kennedy (1994). Differentiating (2.21) in t yields

$$\int_0^t \int_s^t \int_{\mathbb{R}_+} \tau\varkappa(x,y)(1 - e^{-\tau\varkappa(x,y)(t-(x\vee y))})\,\sigma(d\tau)\,dx\,dy = \mu_{t,t} - \mu_{s,t} \tag{2.22}$$

for all $s,t \ge 0$, $s \le t$. Setting $s = 0$ in the latter equation gives

$$\int_0^t \int_0^t \int_{\mathbb{R}_+} \tau\varkappa(x,y)(1 - e^{-\tau\varkappa(x,y)(t-(x\vee y))})\,\sigma(d\tau)\,dx\,dy = \mu_{t,t} - \mu_{0,t}. \tag{2.23}$$

Subtracting (2.22) from (2.23) implies (b). $\square$

Corollary 2.1 *Suppose that the Lévy measure* σ *additionally satisfies*

$$\langle\tau\rangle_\sigma := \int_{\mathbb{R}_+} \tau\,\sigma(d\tau) < \infty. \tag{2.24}$$

Suppose that statement (a) *of Theorem* 2.1 *holds and suppose that the initial term structure* $\{\mu_{0,t},\ t\geq 0\}$ *satisfies*

$$\mu_{0,t} \geq \int_0^t \int_0^t \varkappa(x,y)\,dx\,dy\cdot\langle\tau\rangle_\sigma, \qquad t\geq 0. \tag{2.25}$$

Then the forward rate process $\{F_{s,t},\ 0\leq s\leq t<\infty\}$ *and the spot rate process* $\{R_t,\ t\geq 0\}$ *take on non-negative values* ν*-a.s.*

Proof. By (2.5) and Theorem 2.1, we get

$$F_{s,t} = \mu_{0,t} - \int_0^t \int_0^s \int_{\mathbb{R}_+} \tau\varkappa(x,y)e^{-\tau\varkappa(x,y)(t-(x\vee y))}\,\sigma(d\tau)\,dx\,dy + \widetilde{X}_{s,t}, \qquad t\geq 0,\ 0\leq s\leq t,$$

where

$$\widetilde{X}_{s,t} := X_{s,t} + \int_0^t \int_0^s \varkappa(x,y)\,dx\,dy\cdot\langle\tau\rangle_\sigma.$$

Under condition (2.24), the measure ν is concentrated on the set of all signed measures of the form $\sum_{n=1}^\infty \tau_n\delta_{(x_n,y_n)}(dx\,dy) - \langle\tau\rangle_\sigma\,dx\,dy$, where δ_a denotes the Dirac measure with mass at a, $\tau_n\in\operatorname{supp}\sigma$, $n\in\mathbb{N}$, and $\{(x_n,y_n)\}_{n=1}^\infty$ is a locally finite set in $\mathbb{R}^2$, see, e.g., Lytvynov (2003). Therefore, by (2.4), $\widetilde{X}_{s,t}$ takes on non-negative values ν-a.s. Furthermore, it follows from (2.25) that

$$\mu_{0,t} - \int_0^t \int_0^s \int_{\mathbb{R}_+} \tau\varkappa(x,y)e^{-\tau\varkappa(x,y)(t-(x\vee y))}\,\sigma(d\tau)\,dx\,dy \geq 0, \qquad t\geq 0,\ 0\leq s\leq t,$$

from where the statement follows. $\square$

Let us consider two examples of a measure ν satisfying the assumptions of Theorem 2.1 and Corollary 2.1.

Example 1. (*Poisson sheet*) We take as ν the centered Poisson measure π_z with intensity parameter $z>0$, see, e.g., Hida (1970). The Lévy measure σ has now the form $z\delta_1$. Thus, the Fourier transform of π_z is given by

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\,\pi_z(d\omega) = \exp\left[\int_{\mathbb{R}^2} (e^{i\varphi(x,y)} - 1 - i\varphi(x,y))\,z\,dx\,dy\right], \qquad \varphi\in\mathcal{D}.$$

We set $\varkappa(x,y)\equiv 1$. Then, $X_{s,t}$ given by (2.4) with the underlying probability measure $\nu=\pi_z$ is, by definition, a Poisson sheet, and for each fixed $t>0$, $\{X_{s,t},\ 0\leq s\leq t\}$ is

a centered Poisson process with intensity parameter tz. Statement (b) of Theorem 2.1 now reads as follows:

$$\mu_{s,t} = \mu_{0,t} + z\big((2-s)e^{s-t} - 2e^{-t} - s + st\big).$$

Condition (2.25) now means $\mu_{0,t} \geq zt^2$, $t \geq 0$.

Example 2. (*Gamma sheet*) We take as ν the centered gamma measure γ_z with intensity parameter $z > 0$, see, e.g., Lytvynov (2003). The Lévy measure σ on $\mathbb{R}_+$ has the form

$$\sigma(d\tau) = \frac{e^{-\tau}}{\tau}\, z\, d\tau.$$

The Fourier transform of γ_z may be written as follows:

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\, \gamma_z(d\omega) = \exp\left(-\int_{\mathbb{R}^2} \big(\log(1-i\varphi(x,y))+\varphi(x,y)\big)\, z\, dx\, dy\right), \qquad \varphi \in \mathcal{D},\ |\varphi| < 1.$$

We set $\varkappa(x,y) \equiv 1$. Then, $X_{s,t}$ given by (2.4) with the underlying probability measure $\nu = \gamma_z$ is, by definition, a gamma sheet, and for each $t > 0$, $\{X_{s,t},\ 0 \leq s \leq t\}$ is a centered gamma process with intensity parameter tz. Statement (b) of Theorem 2.1 now reads as follows:

$$\mu_{s,t} = \mu_{0,t} + z\left(st + 2s + 2(1+t)\log\left(\frac{1+t-s}{1+t}\right) - s\log(1+t-s)\right).$$

Condition (2.25) means $\mu_{0,t} \geq zt^2$, $t \geq 0$.

It is possible to construct a model of forward interest rates which unifies the approach of Kennedy (1994) to modeling the forward interest rate with our approach. Indeed, consider $F_{s,t}$ in the form

$$F_{s,t} = \mu_{s,t} + X_{s,t} + Y_{s,t}, \qquad 0 \leq s \leq t, \tag{2.26}$$

where $\mu_{s,t}$ and $X_{s,t}$ are as in formula (2.5) (thus, as in our approach) and $Y_{s,t}$ is a centered continuous Gaussian random field that is independent of $X_{u,v}$, $0 \leq u \leq v < \infty$, and has covariance

$$\mathrm{Cov}(Y_{s_1,t_1}, Y_{s_2,t_2}) = c(s_1 \wedge s_2, t_1, t_2), \qquad 0 \leq s_i \leq t_i,\ i = 1, 2,$$

with a function c satisfying $c(0, t_1, t_2) \equiv 0$ (as in Kennedy's approach).

The following theorem may be easily proved by combining the proof of Theorem 1.1 in Kennedy (1994) and the proof of Theorem 2.1.

Theorem 2.2 *Theorem* 2.1 *remains valid for the forward rates* $\{F_{s,t},\ 0 \leq s \leq t < \infty\}$ *given by* (2.26) *if we set the deterministic term* $\mu_{s,t}$ *in statement* (b) *to be*

$$\mu_{s,t} = \mu_{0,t} + \int_0^t \int_0^s \int_{\mathbb{R}_+} \tau\varkappa(x,y)(1 - e^{-\tau\varkappa(x,y)(t-(x\vee y))})\, \sigma(d\tau)\, dx\, dy + \int_0^t c(s \wedge u, u, t)\, du$$

for all $s, t \geq 0$, $s \leq t$.

References

Adler, R.J.: The geometry of random fields. Chichester: John Wiley & Sons 1981

Berezansky, Yu.M., Sheftel, Z.G., Us, G.F.: Functional analysis, Vol. 2. Basel: Birkhäuser Verlag 1996

Björk, T., Christensen, B.J.: Interest rate dynamics and consistent forward rate curves. Math. Finance **9**, 323–348 (1999)

Björk, T., Di Masi, G., Kabanov, Yu., Runggaldier, W.: Towards a general theory of bond markets. Finance and Stochast. **1**, 141–174 (1997a)

Björk, T., Kabanov, Yu., Runggaldier, W.: Bond market structure in the presence of marked point processes. Math. Finance **7**, 211–239 (1997b)

Gel'fand, I.M., Vilenkin, N.Ya.: Generalized functions, Vol. 4. Applications of harmonic analysis. New York: Academic Press 1964

Heath, D.C., Jarrow, R.A., Morton, A.: Bond pricing and the term structure of interest rates: A discrete time approximation. J. Financial Quant. Anal. **25**, 419–440 (1990)

Heath, D.C., Jarrow, R.A., Morton, A.: Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. Econometrica **60**, 77–105 (1992)

Hida, T.: Stationary stochastic processes. Princeton: Princeton University Press 1979

Ho, T., Lee, S.: Term structure movements and pricing interest rate contingent claims. J. Finance **1**, 1011–1029 (1986)

Kennedy, D.P.: The term structure of interest rates as a Gaussian random field. Math. Finance **4**, 247–258 (1994)

Kennedy, D.P.: Characterizing Gaussian models of the term structure of interest rates. Math. Finance **7**, 107–118 (1997)

Kusuoka, S.: Term structure and SPDE. In: S. Kusuoka and T. Maruyama (eds.): Advances in Mathematical Economics, Vol. 2. Tokyo: Springer 2000, pp. 67–85

Lütkebohmert, E.: Endlich dimensionale Darstellungen für das erweiterte Zinsmodell von Heath, Jarrow und Morton. Diploma Thesis, Bonn: Bonn University 2002

Lytvynov, E.: Orthogonal decompositions for Lévy processes with an application to the gamma, Pascal, and Meixner processes. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6**, 73–102 (2003)

Mahnig, A.: Modellierung der Zinsstrukturen durch ein Poisson Sheet. Diploma Thesis, Bonn: Bonn University 2002

Popovici, S.A.: Modellierung von Zinsstrukturkurven mit Hilfe von stochastischen partiellen Differentialgleichungen. Diploma Thesis, Bonn: Bonn University 2001

Yalovenko, I.: Modellierung des Finanzmarktes und unendlich dimensionale stochastische Prozesse. Diploma Thesis, Bochum: Bochum University 1998